A note on the characteristic rank of a smooth manifold

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ABSTRACT. This paper presents some results, using the characteristic rank recently introduced by the second named author, on those smooth manifolds which can serve as total spaces of smooth fibre bundles with fibres totally non-homologous to zero with respect to $\mathbb{Z}_2$. As the main results, first, some upper and lower bounds for the characteristic rank of those total spaces which need not be null-cobordant are derived; then, bounds for the characteristic rank of null-cobordant total spaces are deduced. Examples are shown, where the upper and lower bounds coincide; thus these bounds cannot be improved in general. All examples of manifolds considered are homogeneous spaces.

1 Introduction

Our aim in this note is to present some results on those smooth manifolds which can serve as total spaces of smooth fibre bundles. More precisely, we mainly shall deal with some situations, where a new homotopy invariant of smooth closed manifolds called the characteristic rank, introduced by the second named author in [4], brings an interesting piece of information. For this, we concentrate on smooth fibre bundles with fibres totally non-homologous to zero: given a smooth fibre bundle $p : E \longrightarrow B$ with total space E, base space B and fibre F we recall (see, e.g., [7, p. 124]) that F is said

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to be totally non-homologous to zero (in E) with respect to a given coefficient ring R if the fibre inclusion $i : F \to E$ induces an epimorphism, $i^* : H^*(E; R) \to H^*(F; R)$, in cohomology.

In the sequel, we shall always understand $R = \mathbb{Z}_2$ and write just $H^i(X)$ instead of $H^i(X; \mathbb{Z}_2)$ for the ith $\mathbb{Z}_2$-cohomology group of X. In addition, all manifolds, thus also the total spaces, fibres and base spaces of smooth fibre bundles, will be (supposed to be, even if we do not mention it explicitly) smooth, connected and closed; all examples of manifolds considered will be homogeneous spaces. Note that for our purposes we may take any fibre $F_b = \{x \in E; p(x) = b\}$, where $b \in B$, in the role of F mentioned above, since B is path connected.

One may think of various types of conditions which must be satisfied by the $\mathbb{Z}_2$-cohomology of the total space E of any smooth fibre bundle $p : E \to B$ with base B and fibre F, if F should be totally non-homologous to zero in E. Among the well-known ones are (see [7], about the Leray-Hirsch theorem), for instance, that the $\mathbb{Z}_2$-Poincaré polynomial $P(E; t) = \sum_i \dim_{\mathbb{Z}_2} H^i(E)t^i$ must be the product $P(B; t)P(F; t)$, that the induced homomorphism $p^* : H^*(B) \to H^*(E)$ must be a monomorphism or that the kernel of $i^* : H^*(E) \to H^*(F)$, where $i : F \to E$ is the fibre inclusion, must be the ideal generated by $p^*(H^+(B))$, where $H^+(B) = \sum_{i>0} H^i(B)$.

For specific manifolds in the role of F or B, we may be able to derive specific conditions. To give a not so well known (as compared to, for instance, spheres or projective spaces) example: for the real Grassmann manifold $G_{2^s+4,4} \cong O(2^s+4)/(O(4)\times O(2^s))$ ($s \geq 3$), consisting of 4-dimensional vector subspaces in $\mathbb{R}^{2^s+4}$, one calculates ([1]) that the height of the third Stiefel-Whitney class $w_3 \in H^3(G_{2^s+4,4})$ of the canonical 4-plane bundle over $G_{2^s+4,4}$ is equal to $2^s + 1$; in other words, we have $w_3^{2^s+1} \neq 0$, but $w_3^{2^s+2} = 0$. Therefore if E were the total space of a smooth fibre bundle over B with $G_{2^s+4,4}$ as fibre F, this fibre being totally non-homologous to zero, then there must exist an element $x \in H^3(E)$ such that $i^*(x) = w_3$, $x^{2^s+1} \neq 0$, and x^{2^s+2} must lie in the ideal generated by $p^*(H^+(B))$.

As is known (see, for instance, [6]), the Stiefel-Whitney characteristic classes $w_i(M) \in H^i(M)$ of a (smooth, closed, connected) manifold M are identified with the Stiefel-Whitney classes of its tangent bundle TM, thus $w_i(M) = w_i(TM)$. These characteristic classes are crucial in studying several fundamental properties of M. Most notably, M is orientable if and only if $w_1(M) = 0$. But, as already indicated in [4], it turns out that the degree up to which the cohomology algebra $H^*(M)$ is generated by the Stiefel-Whitney classes of M also carries useful information.

More precisely, in [4] the second named author defined the characteristic rank, briefly $\operatorname{charrank}(M)$, of a d-dimensional manifold M, to be the largest integer k, $0 \leq k \leq d$, such that each element of the cohomology group $H^j(M)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of M. For instance, if M is orientable and $H^1(M) \neq 0$, then we have $\operatorname{charrank}(M) = 0$. For more results on the values of characteristic rank, see [4].

The usefulness of the characteristic rank is already clear from the following theorem. By $\operatorname{cup}(M)$ we denote the $\mathbb{Z}_2$-cup-length of the manifold M, hence the maximum of all numbers c such that there exist, in positive degrees, cohomology classes $a_1, \ldots, a_c \in H^*(M)$ such that their cup product $a_1 \cup \cdots \cup a_c$ is nonzero. In addition, let r_M denote the smallest number such that the reduced cohomology group $\tilde{H}^{r_M}(M)$ does not vanish (we note that $0 < r_M \leq d$ since M is a connected d-dimensional manifold).

Theorem 1.1. (Korbaš [4, Theorem 1.1]) *Let M be a closed, smooth, connected, d-dimensional, unorientedly null-cobordant manifold. Then we have that*

$$\operatorname{cup}(M) \leq 1 + \frac{d - \operatorname{charrank}(M) - 1}{r_M}. \tag{1}$$

In the following section, we shall show that a new type of numerical condition which is satisfied by the total space E, if F is totally non-homologous to zero in E, can be obtained by using the characteristic rank. As the main results, we shall first derive (in Theorem 2.1) some upper and lower bounds for the characteristic rank of those total spaces which need not be null-cobordant (zero-cobordant), and then we shall deduce (in Theorem 2.2) bounds for the characteristic rank of null-cobordant total spaces. In addition, (infinitely many) non-trivial fibre bundles will be exhibited for which our upper and lower bounds coincide; thus these bounds cannot be improved in general.

2 The characteristic rank for smooth fibre bundles with fibre totally non-homologous to zero

2.1 General total spaces

In this subsection, as the first of our main results, we give some bounds for the characteristic rank of the total space of any smooth fibre bundle with fibre totally non-homologous to zero; the total space need not be null-cobordant.

Theorem 2.1. *Let $p : E \to B$ be a smooth fibre bundle with fibre F totally non-homologous to zero. Then we have that* (*if* $\mathrm{charrank}(E) \le \dim(F)$, *then*)

$$\min\{r_B, r_F\} - 1 = r_E - 1 \le \mathrm{charrank}(E) \le \mathrm{charrank}(F).$$

Proof. It is clear that $\mathrm{charrank}(E) \ge r_E - 1$. Since for the Poincaré polynomials we now have $P(E;t) = (1 + t^{r_E} + \dots) = P(B;t)P(F;t) = (1 + t^{r_B} + \dots)(1 + t^{r_F} + \dots)$, we see that $r_E = \min\{r_B, r_F\}$, and so it is true that $\mathrm{charrank}(E) \ge r_E - 1 = \min\{r_B, r_F\} - 1$. [Of course, in detail, $P(E;t) = 1 + t^{r_E} + \sum_{i \ge r_E, e_i \ge 0} e_i t^i$, and similarly for $P(B;t)$ and $P(F;t)$.]

It remains to prove that $\mathrm{charrank}(E) \le \mathrm{charrank}(F)$. Take any cohomology class $x \in H^k(F)$ with $k \le \mathrm{charrank}(E)$. Since $i^* : H^*(E) \to H^*(F)$ is an epimorphism, there exists some $y \in H^k(E)$ such that $i^*(y) = x$. Thanks to the fact that $k \le \mathrm{charrank}(E)$, we have that $y = Q(w_1(E), w_2(E), \dots)$ for some polynomial Q.

For any smooth fibre bundle we have $TE \cong p^*(TB) \oplus \kappa$, where κ is the vector bundle along the fibres (so that $i^*(\kappa) \cong TF$). As a consequence, for the Stiefel-Whitney characteristic classes we have $i^*(w_t(E)) = w_t(F)$ for all t, thus implying that

$$x = i^*(y) = i^*(Q(w_1(E), w_2(E), \dots)) = Q(w_1(F), w_2(F), \dots).$$

This finishes the proof. □

Remark 2.1. For any (smooth, closed, connected) manifolds M and N, we have two obvious trivial fibre bundles with the same total space $M \times N$. As a special case of the preceding theorem, we obtain that

$$\min\{r_M, r_N\} - 1 \le \mathrm{charrank}(M \times N)$$

and

$$\mathrm{charrank}(M \times N) \le \min\{\mathrm{charrank}(M), \mathrm{charrank}(N)\}.$$

Remark 2.2. As a consequence of the fact that $i^*(w_t(E)) = w_t(F)$ for all t, any smooth fibre bundle $p: E \to B$ with fibre F such that $\mathrm{charrank}(F) = \dim(F)$ has fibre totally non-homologous to zero; see [3] for further details. Hence Theorem 2.1 applies, in particular, to all fibre bundles such that $\mathrm{charrank}(F) = \dim(F)$.

Remark 2.3. The following example shows one of possible uses of Theorem 2.1 and testifies that, in general, the bounds for $\mathrm{charrank}(E)$ given by Theorem 2.1 cannot be improved.

Example 2.1. We calculate the characteristic rank for the complex flag manifolds $F(1,1,n-2) \cong U(n)/(U(1) \times U(1) \times U(n-2))$. We recall that $F(1,1,n-2)$ may be interpreted to consist of triples (S_1, S_2, S_3), where S_i are mutually orthogonal vector subspaces in $\mathbb{C}^n$ such that $\dim_{\mathbb{C}}(S_1) = \dim_{\mathbb{C}}(S_2) = 1$ and $\dim_{\mathbb{C}}(S_3) = n-2$. Then one has a smooth fibre bundle over the complex Grassmann manifold $\mathbb{C}G_{n,2} \cong U(n)/(U(2)\times U(n-2))$ (consisting of complex 2-dimensional vector subspaces in $\mathbb{C}^n$), $p: F(1,1,n-2) \to \mathbb{C}G_{n,2}$, $p(S_1,S_2,S_3) = S_1 \oplus S_2$. One can see in several ways (for instance, by applying [7, Ch. 3, Lemma 4.5]) that the fibre, the complex projective space $\mathbb{C}P^1$ (known to be diffeomorphic to the 2-dimensional sphere S^2), is totally non-homologous to zero with respect to $\mathbb{Z}_2$. Of course, we have $\mathrm{charrank}(\mathbb{C}P^1) = 1$. Now for $E = F(1,1,n-2)$, $B = \mathbb{C}G_{n,2}$, $F = \mathbb{C}P^1$, we have $r_B = r_F = 2$, hence the lower and upper bounds given by Theorem 2.1 coincide, and we obtain that $\mathrm{charrank}(F(1,1,n-2)) = 1$.

2.2 Null-cobordant total spaces

For any null-cobordant manifold E, one has (cf. [4]) $\mathrm{charrank}(E) < \dim(E)$. For the characteristic rank of such a manifold, if it serves as the total space of a smooth fibre bundle with fibre totally non-homologous to zero, we now derive, as the second of our main results, the following.

Theorem 2.2. *Let $p: E \to B$ be a smooth fibre bundle with E null-cobordant and with fibre F totally non-homologous to zero. Then we have that* (*if* $\mathrm{charrank}(E) \le \dim(F)$, *then*)

$$\min\{r_B, r_F\} - 1 = r_E - 1 \le \mathrm{charrank}(E) \le \min\{u_{B,F}, \mathrm{charrank}(F)\},$$

where $u_{B,F} = \dim(B) + \dim(F) - 1 - \min\{r_B, r_F\}(\mathrm{cup}(B) + \mathrm{cup}(F) - 1)$.

Proof. We have now by Horanská and Korbaš [2, Lemma, p. 25] that

$$\mathrm{cup}(E) \geq \mathrm{cup}(B) + \mathrm{cup}(F).$$

At the same time, by Theorem 1.1 we have the inequality (1) for $M = E$. Thus we obtain the inequality

$$r_E \mathrm{cup}(B) + r_E \mathrm{cup}(F) \leq r_E + \dim(E) - \mathrm{charrank}(E) - 1,$$

and this gives the upper bound for charrank(E) stated in the theorem, if we take into account that $r_E = \min\{r_B, r_F\}$, $\dim(E) = \dim(B) + \dim(F)$ and that, in addition, by Theorem 2.1 we have the inequality $\mathrm{charrank}(E) \leq \mathrm{charrank}(F)$. The lower bound is the same as in Theorem 2.1. This finishes the proof. □

The following example is a non-trivial application of Theorem 2.2 and also gives evidence that, in general, the bounds for charrank(E) given by Theorem 2.2 are sharp.

Example 2.2. We again calculate, this time in a different way (as compared to Example 2.1), the characteristic rank for the complex flag manifolds $F(1, 1, n-2)$. We now take a smooth fibre bundle over the complex projective space $\mathbb{C}P^{n-1}$, $p : F(1, 1, n-2) \to \mathbb{C}P^{n-1}$, $p(S_1, S_2, S_3) = S_1$. Its fibre, the complex projective space $\mathbb{C}P^{n-2}$, is totally non-homologous to zero with respect to $\mathbb{Z}_2$ (this can be seen in several ways; for instance, apply [7, Ch. 3, Lemma 4.5]). There is an obvious smooth fixed point free involution on $F(1, 1, n-2)$, interchanging S_1 and S_2 for every $(S_1, S_2, S_3) \in F(1, 1, n-2)$; in other words, the group $\mathbb{Z}_2$ acts smoothly and without fixed points on $F(1, 1, n-2)$. As a consequence (cf. [5]), the flag manifold $F(1, 1, n-2)$ is null-cobordant. We have $\mathrm{cup}(\mathbb{C}P^k) = k$ (see for instance [8, Theorem 15.33]). Now for $E = F(1, 1, n-2)$, $B = \mathbb{C}P^{n-1}$, $F = \mathbb{C}P^{n-2}$, we have $r_B = r_F = 2$ and $u_{B,F} = 2n-2+2n-4-1-2(n-1+n-2-1) = 1$, hence the lower and upper bounds given by Theorem 2.2 coincide, and we obtain that $\mathrm{charrank}(F(1, 1, n-2)) = 1$.

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References

[1] Balko, Ľ.: *The height of the third canonical Stiefel-Whitney class of the Grassmann manifold of four-dimensional subspaces of Euclidean space.* (in Slovak)

MSc.-Thesis, Faculty of Mathematics, Physics, and Informatics, Comenius University, Bratislava 2008.

[2] Horanská, Ľ., Korbaš, J.: *On cup products in some manifolds*, Bull. Belg. Math. Soc. - Simon Stevin **7** (2000), 21-28.

[3] Korbaš, J.: *On fibrations with Grassmannian fibers*, Bull. Belg. Math. Soc. - Simon Stevin **8** (2001), 119-130.

[4] Korbaš, J.: *The cup-length of the oriented Grassmannians vs a new bound for zero-cobordant manifolds*, Bull. Belg. Math. Soc. - Simon Stevin **17** (2010), 69-81.

[5] Sankaran, P., Varadarajan, K.: *Group actions on flag manifolds and cobordism*, Canadian J. Math. **45** (1993), 650-661.

[6] Milnor, J., Stasheff, J.: *Characteristic Classes*, Ann. Math. Stud. 76, Princeton Univ. Press, Princeton, N. J. 1974.

[7] Mimura, M., Toda, H.: *Topology of Lie Groups. Part I.* Translations of Math. Monographs 91, American Mathematical Society, Providence, RI 1991.

[8] Switzer, R.: *Algebraic Topology – Homotopy and Homology*, Springer, Berlin 1975.

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